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Skin Friction of Slender Cones in Hypersonic Flow

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The transverse-curvature effect on the skin friction of slender, circular cones is investigated employing a momentum-integral technique A relation between a suitable transverse-curvature parameter and the hypersonic viscous interaction parameter is discussed

1 Introduction

APPLICATION of the Mangler transformation reduces the axisymmetric boundary-layer problem to the computation of a two-dimensional boundary layer The trans formation becomes possible through the assumption that the radial coordinate, wherever it occurs explicitly in the boundary-layer equations, may be replaced by the local body radius In other words, it is assumed that the body radius is large compared with the boundary-layer thickness This assumption may become invalid, however, for slender bodies in hypersonic flight at high altitudes, such as the case of reentry missiles It will then be necessary to solve the boundary-layer equations with the additional transverse-curvature terms Unfortunately, these equations, in general, do not yield similar solutions In the regime of vanishing transverse-curvature effects, however, asymptotic solutions have been obtained by Probstein and Elliott [1] In the following, a momentum-integral technique is discussed which has been employed to obtain an approximate solution in the regime where transverse-curvature effects are neither very small nor very large, and where, consequently, series-expansion techniques fail

2 Analysis

In the absence of pressure variations the momentum loss in the boundary layer is due to friction only, and the momentum theorem requires

$$\frac{d}{dx}\int_0^\infty \rho u(u \quad - u)2\pi r dy = 2\pi r_w \tau_w \qquad (1)$$

where conventional notation has been adopted †

Introduction of the variable

$$\lambda = \int_0^y \frac{\rho}{\rho}\, r dy$$

reduces Eq (1) to the incompressible form

$$\frac{d}{dx}\int_0^\infty \frac{u}{u_e}\left(1 - \frac{u}{u}\right) d\lambda = \frac{r_w \tau_w}{\rho\, u^2} \qquad (2)$$

It can be shown[2] from the governing boundary-layer differential equations that, for a heat-insulating wall and for a gas following the Chapman-Rubesin viscosity law $\mu/\mu \;= C\; T/T$, the velocity profile $u/u \;= f(x,\lambda)$ can be expressed as

$$f(x,\lambda) = \frac{1}{\Phi}\frac{\partial f(x,0)}{\partial \lambda}\left[\Phi\lambda - \frac{1}{2}\Phi^2\lambda^2 + \frac{1}{3}\Phi^3\lambda^3 + O(\lambda^4)\right]$$

where

$$\Phi = \frac{2\cos\theta}{r_w^2}\frac{T_w}{T}$$

The angle θ is the local inclination of the body surface with respect to the freestream (in the case of a cone, θ is the semivertex angle)

The foregoing expansion indicates that a profile proportional to $\ln(1 + \Phi\lambda)$ will give a fairly accurate estimate of the actual boundary-layer flow near the body surface, because such a profile will be correct up to the fourth order in $\Phi\lambda$ It further can be shown that, for a gas following a power viscosity law instead of the linear Chapman-Rubesin relation, the logarithmic profile is correct up to only the third order in $\Phi\lambda$ This will also be the case for a heat-conducting surface If we choose the profile‡

$$f(x,\lambda) = [1/\alpha(x)]\ln(1 + \Phi\lambda)$$
$$f(x,\lambda) = 1 \text{ for } \lambda \geqslant (1/\Phi)\,(e^\alpha - 1) \qquad (3)$$

the following differential equation for $\alpha(x)$ is obtained upon introducing Eq (3) into Eq (2):

$$\frac{d}{dx}\left[\frac{r_w^2}{\cos\theta}\left\{\frac{2}{\alpha^2}\,(1 - e^\alpha) + \frac{1}{\alpha}\,(1 + e^\alpha)\right\}\right] = \frac{4\mu_w\cos\theta}{\rho_w u_e \alpha} \qquad (4)$$

Equation (4) is still applicable to a body of revolution of arbitrary shape as long as pressure variations along the body are negligible Integration of Eq (4) in the particular case of a cone leads to the following expression for the local skin-friction coefficient[2]:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho\, u^2} = [1 + F(t)]\left(\frac{C\nu_e}{u\, x}\right)^{1/2} \qquad (5)$$

where

$$t = \cot\theta\,\frac{T_w}{T}\left(\frac{C\nu}{u\, x}\right)^{1/2} \qquad (6)$$

The parameter t is the transverse-curvature parameter of Probstein and Elliott, multiplied by $3^{1/2}T_w/T$ The functional dependence $F(t)$ has been computed from the numerical solution for α and is given in Fig 1 Furthermore, the following asymptotic solutions for $F(t)$ can be derived from Eq (4):

$$F(t \to 0) = \tfrac{4}{5}t - \tfrac{1}{25}t^2 + \qquad (7)$$

$$1 + F(t \to \infty) = 4t[\ln^{-1}(4t^2) - \ln^{-3}(4t^2) + (-\tfrac{7}{2})\ln^{-4}(4t^2) + \quad] \qquad (8)$$

Equation (7) applies to the regime of vanishing curvature effects far downstream of the cone apex, whereas Eq (8) applies to the immediate vicinity of the cone apex in the regime of very strong curvature effects

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† The coordinates x and y, respectively, are parallel and normal to the body surface with the origin at the body apex or leading edge; r is the radial coordinate The subscripts e and w, respectively, refer to the outer edge of the boundary layer and the body surface

‡ For incompressible flow and $\theta = 0$, Eq (3) reduces to the profile employed by Glauert and Lighthill[3] in their analysis of the boundary layer along a slender, circular cylinder

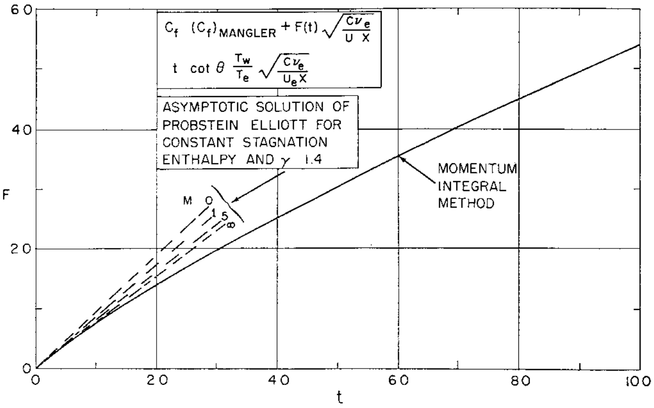

Fig 1 Transverse-curvature correction on the Mangler skin-friction coefficient for a cone

3 Discussion

It now is possible to arrive at an estimate of the accuracy of the present method, at least in the regime of weak curvature effects, by comparing Eq (7) with the asymptotic expansion of Probstein and Elliott [1] If we assume a constant stagnation enthalpy throughout the boundary layer (which corresponds to an insulated wall and a Prandtl number of unity), it follows from Eq (7) for the local skin-friction coefficient that

$$C_f = \left(\frac{C\nu_e}{u\,x}\right)^{1/2} + 0\,80\,\cot\theta\left(1 + \frac{\gamma - 1}{2}\,Me^2\right)\frac{C\nu_e}{u_e x} + \tag{9}$$

The corresponding solution of Probstein and Elliott is

$$C_f = 1\,15\left(\frac{C\nu_e}{u\,x}\right)^{1/2} + \cot\theta\left(0\,95 + 0\,77\,\frac{\gamma - 1}{2}\,Me^2\right)\frac{C\nu_e}{u_e x} + \tag{10}$$

The leading term of Eq (9) is the flat-plate result as obtained with a straight-line profile, multiplied by the Mangler factor $3^{1/2}$ This term is in error by about 13% compared with the leading term of Eq (10), the latter being the Chapman-Rubesin flat-plate expression multiplied by the same Mangler factor The deviation of the incompressible part of the second term in Eq (9) amounts to almost 16% However, for increasing Mach numbers, the compressible part of the second term, being in error by about 4%, will gradually dominate

In Fig 1 the function $F(t)$ is compared with the asymptotic solution of Probstein and Elliott for the case of constant stagnation enthalpy and $\gamma = 1\,4$

For hypersonic flow over a very slender insulated cone, it follows approximately from Eq (6) that

$$t \propto \frac{Me^2}{\theta}\left(\frac{C\nu_e}{u\,x}\right)^{1/2} = \frac{\chi}{K} \tag{11}$$

where $\chi = C^{1/2}M^{\;3}/(Re_x)^{1/2}$ is the local hypersonic viscous interaction parameter, and $K = M_e\theta$ is the local hypersonic similarity parameter In many cases of practical interest, K will be of the order of unity

It should be kept in mind, however, that the present analysis is based on the assumption of constant pressure throughout the boundary layer Consequently, the effect of the self-induced pressure distribution on the boundary-layer development has not been accounted for

The momentum-integral analysis sketched in the preceding paragraphs is readily extended[2] to include (slender) bodies of revolution of the type $r_w \propto x^n$, provided that pressure gradients along the body are neglected In that case we employ the more general transverse curvature parameter

$$t = \frac{\cos\theta}{r_w}\,\frac{T_w}{T_e}\left(\frac{C\,\nu_e\,x}{u_e}\right)^{1/2}$$

and we find, as in Eq (5), that the local skin-friction coefficient equals its Mangler value multiplied by a function which depends on the parameter t only It is further easily verified that, for a slender insulated power-law body, relation (11) still holds true

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Heat Transfer Due to Hydromagnetic Channel Flow with Conducting Walls

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Nomenclature

μ	= magnetic permeability
σ	= electrical conductivity of the fluid
$\mathbf{H}$	= magnetic field vector
H_y	= applied magnetic field
$\mathbf{J}$	= current density
$\mathbf{E}$	= electric field
$\mathbf{q}$	= velocity of the fluid
ρ	= density of the fluid
c	= velocity of light
η	= electrical diffusivity = $(1/\sigma)c^2/(4\pi\mu)$
ν	= kinematic viscosity
p_m	= magnetic Prandtl number = η/ν
a	= Alfven's wave velocity = $(\mu\,H_y{}^2/4\pi\rho)^{1/2}$
R_m	= magnetic Reynold s number = aL/ν
M	= Hartmann number = $aL(\eta\nu)^{-1/2}$
σ_{w1},σ_{w2}	= conductivity of the lower and upper walls
h_1,h_2	= thickness of the lower and upper walls
ϕ_1,ϕ_2	= electrical conductance parameters of the lower and upper walls
c_p	= specific heat
K	= thermal conductivity of the fluid
E_m	= magnetic Eckert's number = $a^2/c_p\theta_1$
P	= Prandtl number = $\mu c_p/K$
L	= channel half width

THE problem of the two-dimensional flow of an incompressible, viscous, and electrically conducting fluid through a channel formed by two parallel nonconducting walls subjected to the action of uniform transverse magnetic field in the presence of heat transfer was considered in Refs 1–3 The first reference contained a solution for a uniform heat flux without considering viscous dissipation However, in the second reference, the effect of viscous dissipation was taken into account In Ref 3 the problem

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